**Unit 7: Modeling with Autonomous Differential Equations**

Goals/Rationale

Graphs of autonomous differential equations (that is, graphs of dy/dt vs. y) can be extremely useful tools for determining qualitatively correct graphs of solutions and for analyzing the long term behavior of all solutions, including the number and nature of equilibrium solutions (we refer to this as the structure of the solution space). A primary goal of this unit, therefore, is to motivate the need for dy/dt vs. y graphs and to help students develop the ability to interpret and use such graphs to determine the structure of the solution space. Population contexts are used to motivate the need for such graphs and to give students practice with modeling.

**Page 7.1**

Implementation Notes

This is the first time that students are asked to create a differential equation based on data. Note, they previously created a DE from physical laws (rate in – rate out) and they modified DEs based on different a change of assumptions about the phenomenon (e.g., resources are not unlimited). Consider getting this problem started by situating this problem within these three ways of creating differential equations.

*Problem 1* - Students will likely fill in the third column by taking a ratio of dC/dt. One question that invariably comes up is where to put the first ratio – in the same row as time 0, in the row with time 2, in between? Note: it is better to either use the first or second row (rather than splitting the difference) because otherwise graphing the data will be problematic in number 2. Consider asking questions such as:

* Why did you decide to put the value of dC/dt where you did? Did anyone do it a different way?
* If you wanted to match Euler’s method, where should you put the value of dC/dt and why?

*Problem 2* - Give students several minutes to think about what dC/dt should explicitly depend on and then collect their ideas and reasons. Do not move on to the next problem until students have figured out that the rate of change does not explicitly depend on time. If they have difficulty with this consider asking the following:

* What if the experiment was repeated an hour later or at the same time the next day? Do you expect the data collected to be different?

*Problem 3* - Students are asked to sketch a graph of dC/dt vs C that fits their data. Students often do not graph a straight line and rarely do they extend their graph above the horizontal axis. Typical graphs will either be straight below the horizontal axis or curving toward the horizontal axis from below. It is not necessary to come to consensus at this point or even that students get a qualitatively correct graph because in the next problem students will use a GeoGebra app to enter their data and curve fit the data.

**Page 7.2**

Implementation Notes

* Do not hand out problem 5 until problem 4 has been completed and discussed.

**Page 7.3**

Implementation Notes

*Problem 5 -* Although we used the equation dC/dt = -.4C +28 (which more or less corresponds to a data table consistent with Euler’s method), there is nothing coming up that relies of this particular equation, so feel free to use students’ equation instead of this one.

One thing that students typically need to discuss is why it makes sense to extend the graph of dC/dt vs. C above the C axis. Possible discussion questions:

* Why does this graph of dC/dt vs C not make sense?



* For a linear graph: Which portion of the graph corresponds to a glass of iced tea and why? What does the differential equation predict will happen to the temperature of the iced tea?
* What portion of the graph corresponds to a glass of water that is at room temperature?

*Problem 6 -* Here are three possible ways that students might come up with to argue against the given graph.

* This is not possible because the graph means that once the coffee temperature cools down to nearly room temperature then it gets colder than room temperature, then warmer then room temperature, then colder than room temperature, etc.
* The graph contradicts the slope field, which has arrows above 70 all with negative slope and arrows below 70 all with positive slope. Also, the slope field has vectors with zero slope at 70, which also contradicts the given graph.
* The differential equation satisfies the conditions of the uniqueness theorem so the graph would never touch or cross the equilibrium solution graph of C(t) = 70.

Consider recording student thinking with a phase line. Then ask students:

* While there is no phase line shown on the graphs of dC/dt vs C, where does it makes sense to draw one in on this graph?

**Page 7.4 – Population Growth: Limited Resources**

Implementation Notes

*Problem 7* – Note that the data given here is different that the previous set of data and estimating dP/dt is done by subtracting values in the table. Possible discussion questions:

* Is it better to subtract the value in the second column from the value in the first or the other way around and why?

*Problem 8* – Students could use excel to plot the data or just graph rough sketch by hand.

**Pages 7.5 – 7.6 – Analyzing Graphs of Autonomous Differential Equations**

Implementation Notes

*Problem 9* – This graph is intended to evoke the previous modeling contexts where data was collected and a graph of dN/dt vs N was plotted based on the data. Students are asked to use the graph of dN/dt vs N to determine the long term behavior of solutions with different initial populations. Encourage students to draw a phase line, a rough slope field, and graphs of solution functions. Possible discussion questions:

* What happens to the population is the initial condition is N =0? How can you determine this from the graph of dN/dt vs N?
* If a fellow student missed today’s class, how would you explain to him or her how to “read” the N axis for the graph of dN/dt vs N?

*Problem 10* - Can be relegated to homework if short on time. If time, here is a possible discussion question:

* Given the phase line you just drew and what we know about the differential equation: Suppose y(0) = -1.9, can you find a time t\* such that y(t\*) = -1.5?

**Notes for Personal Reflections on Unit 7**